Mean and variance.

Let x be a discrete random variable then its expected value denoted by $E(x) = \sum_x x\, p(x)$ where $p(x)$ is the corresponding pmf. The variance of x ($var(x)$) is defined as expectation of $(x - E(x))^2$

$$var(x) = E\{(x - E(x))^2\}$$

$x_1, x_2, \ldots \ldots E(x)$

$E[(x_1 - E(x))^2 (x_2 - E(x)^2) \ldots$

and it can be shown that $var(x) = E(x^2) - [E(x)]^2$

where $E(x^2) = \sum x^2\, p(x)$

The mean and variance of a random variable x are denoted by μ and σ_x^2

i.e. $\mu = E(x)$ and $\sigma_x^2 = Var(x)$

The square root of variance is called the standard deviation.

Some properties of Expectation.

1) $E(aX + b) = aE(x) + b$

2) $E[g(x)] = \sum g(x) p(x)$

Q. The probability mass function of a discrete random variable is given below.

$$f(x) = \begin{cases} \frac{kx}{5}, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

find: 1) The value of k.

2) Probability that $x \geq 2$

3) $E(x)$ and $var(x)$

Soln:

1) For a probability mass function

$\sum p(x) = 1$

$$\frac{k \times 0}{5} + \frac{k \times 1}{5} + \frac{k \times 2}{5} + \frac{k \times 3}{5} = 1$$

$$\frac{k + 2k + 3k}{5} = 1 \Rightarrow \frac{6k}{5} = 1 \Rightarrow k = \frac{5}{6}$$

2) $P(X \geq 2) = P(X = 2, 3)$

$= P(X = 2) + P(X = 3)$

$$= \frac{k \times 2}{5} + \frac{k \times 3}{5} = \frac{5k}{5} = k = \frac{5}{6}$$

3) $E(x) = \sum x p(x)$

$$= 0 \times \frac{k \times 0}{5} + 1 \times \frac{k \times 1}{5} + 2 \times \frac{k \times 2}{5} + 3 \times \frac{k \times 3}{5}$$

$$= 0 + \frac{k}{5} + \frac{4k}{5} + \frac{9k}{5} = \frac{14k}{5}$$

$$= \frac{14}{5} \times \frac{5}{6} = \frac{7}{3}$$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 p(x)$

$= 0^2 \times \frac{k \times 0}{5} + 1^2 \times \frac{k \times 1}{5} + 2^2 \times \frac{k \times 2}{5} + 3^2 \times \frac{k \times 3}{5}$

$= 0 + \frac{k}{5} + \frac{8k}{5} + \frac{27k}{5} = \frac{36k}{5} = \frac{\cancel{36}^{6}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{6}} = \underline{\underline{6}}$

$Var(X) = 6 - \left(\frac{7}{3}\right)^2 = 6 - \frac{49}{9} = \frac{54-49}{9} = \frac{5}{9} = \frac{5}{9}$ //

Q) A random variable X has the following probability distribution

x	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	$2k$	0.3	$3k$

1) find k

2) $P(-2 < x < 2)$

3) mean and variance of x.

Soln:

For a probability distribution $\sum p(x) = 1$

$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$0.6 + 6k = 1$

$6k = 0.4$

$k = \frac{0.4}{6} = \frac{0.2}{3} = \frac{1}{15}$ //

2) $P(-2 < x < 2) = P(-1, 0, 1)$

$= P(X=-1) + P(X=0) + P(X=1)$

$= k + 0.2 + 2k$

$= \frac{1}{15} + 0.2 + 2 \times \frac{1}{15} = \frac{1}{5} + 0.2 = \cancel{\frac{5 \cdot 0.2}{5}}$

$= \cancel{\frac{2}{50}} \frac{1}{25}$

$= \frac{1+1}{5} = \frac{2}{5}$

3) Mean $= E(x)$

$= \Sigma x p(x)$

$= (-2) \times 0.1 + (-1)k + 0 \times 0.2 + 1 \times 2k + 2 \times 0.3 + 3 \times 3k$

$= -0.2 - k + 2k + 0.6 + 9k = 10k + 0.4$

$= \frac{10}{15} + 0.4 = \underline{\underline{0.0667}}$

$E(x^2) = \Sigma x^2 p(x)$

$= (-2)^2 \times 0.1 + (-1)^2 \times k + 0^2 \times 0.2 + 1^2 \times 2k + 2^2 \times 0.3$
$+ 3^2 \times 3k$

$= 0.4 + k + 2k + 1.2 + 27k$

$= 30k + 1.6 = \overset{2}{\cancel{30}} \times \frac{1}{\cancel{15}} + 1.6 = 2 + 1.6 = \underline{\underline{3.6}}$

Q. The Probability distribution of a random variable x is

x	0	1	2	3
$P(X=x)$	0.1	0.3	0.4	0.2

If $y = x^2 + x$ find $E(y)$

Soln:

$Y = x^2 + x$

$E(Y) = E(x^2 + x)$

$= \Sigma (x^2+x)\, p(x)$

$= (0^2+0)\, 0.1 + (1^2+1) \times 0.3 + (2^2+2) \times 0.4 + (3^2+3) \times 0.2$

$= 2 \times 0.3 + 6 \times 0.4 + 12 \times 0.2$

$= 0.6 + 2.4 + 2.4 = \underline{5.4}$

$$\begin{array}{r} 2.4 \\ 2.4 \\ 0.6 \\ \hline 5.4 \end{array}$$

Q) Q. The probability distribution of a random variable x is given by $p(x = x) = \frac{k}{2^x}$, $x = 0, 1, 2, 3, 4$ find the value of k when $P(X \neq 3)$.

Soln:

1) $\Sigma p(x) = 1$

$\frac{k}{2^0} + \frac{k}{2^1} + \frac{k}{2^2} + \frac{k}{2^3} + \frac{k}{2^4} = 1$

$k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \frac{k}{16} = 1$

$k\left(\frac{16+8+4+2+1}{16}\right) = 1$

$k = \frac{16}{31}$ //

2) $P(X \neq 3) = P(X = 0, 1, 2, 4)$

$= 1 - P(x = 3)$

$= 1 - \frac{k}{2^3} = 1 - \frac{1}{8} \times \frac{16}{31}^2 = 1 - \frac{2}{31} = \frac{31-2}{31}$

$= \frac{29}{31}$ //

Some Special Distributions

1) Binomial Distribution

A discrete random variable x is said to follow binomial distribution with parameters n and p if its pmf is

$p(x) = {}^nC_x \, p^x q^{n-x}$, where $x = 0, 1, 2, \dots n$ and $p+q=1$

If x follows binomial distribution with parameters n and p we denote this by $B(n,p)$

Interpretation

Consider an experiment with only 2 possible outcomes say "success' and 'failure'. Let p be the probability for success and q be the probability for failure then $p+q=1$. Suppose the experiment is repeated n times Let X denote the no of successes. Then X is a random variable which can takes the value $0, 1, 2, \dots n$

$P(X=x) = P($ getting x successes in n trials$)$

$= {}^nC_x \, p^x q^{n-x}$ which is the binomial distribution

Q). The probability that a batsman scores a centuary in a cricket match is 1/7. Find the probability that out of 5 matches he may score centuary in

1) exactly 2 matches

2) No match

success → score century

$P = P(\text{Scoring a century}) = \frac{1}{7}$

$q = 1 - p = 1 - \frac{1}{7} = \frac{6}{7}$ $n = 5$

Let x denote the no of matches in which the batsman scores centuary then x is a binomial variable with pmf

$$p(x) = 5_{C_x}\left(\frac{1}{7}\right)^x\left(\frac{6}{7}\right)^{5-x} \quad \text{where } x = 0,1,2,3,4,5$$

1) P(exactly 2 matches) = P(X = 2)

$$= 5_{C_2}\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^{5-2}$$

$$= 5_{C_2} \times \frac{1}{49} \times \left(\frac{6}{7}\right)^3$$

$$= \underline{0.12}$$

2) $P(x=0) = {}^5C_0 \times \left(\frac{1}{7}\right)^0 \times \left(\frac{6}{7}\right)^{5-0}$

$= 1 \times \frac{1}{1} \times \frac{6^5}{7^5} = \frac{6^5}{7^6} = \underline{0.46}$

Q. In the Long run 3 ships out of every hundred are sunk. If 10 ships are out what is the probability that a) exactly 6 will arrive safely.

b) at least 6 will arrive safely.

Soln:

a) $p = P(\text{safe arrival}) = \frac{97}{100} = 0.97$

$q = 1 - p = 1 - 0.97 = 0.03 \qquad n = 10$

Let x denote the no of safe arrivals then x is a binomial random variable with pmf

$p(x) = {}^{10}C_x (0.97)^x (0.03)^{10-x}, \quad x = 0, 1, 2, \ldots 10$

$P(\text{exactly 6 safe arrival}) = P(X=6)$

$= {}^{10}C_6 (0.97)^6 (0.03)^4$

$= \underline{1.4168 \times 10^{-4}}$

b) $P(\text{at least } 6) = P(X = 6, 7, 8, 9, 10)$

$= {}^{10}C_6 (0.97)^6 (0.03)^4 \times {}^{10}C_7 (0.97)^7 (0.03)^3$

$\times {}^{10}C_8 (0.97)^8 (0.03)^2 \times {}^{10}C_9 (0.97)^9 (0.03)^9$

$\times {}^{10}C_{10} (0.97)^{10} (0.03)^0$

=

Q. Find the mean and variance of a binomial random variable.

Soln: Let x be a binomial random variable then the pmf of x is

$$p(x) = {}^nC_x\, p^x q^{n-x}, \text{ where } x = 0, 1, 2, \ldots n$$

Mean $= E(X)$

$= \sum x\, p(x)$

$= 0 \times {}^nC_0\, p^0 q^{n} + 1 \times {}^nC_1\, p^1 q^{n-1} + 2 \times {}^nC_2\, p^2 q^{n-2} + 3 \times {}^nC_3\, p^3 q^{n-3} \ldots\ldots + n\, {}^nC_n\, p^n q^{n-n}$

$= npq^{n-1} + \cancel{2} \times \frac{n(n-1)}{1 \times \cancel{2}} p^2 q^{n-2} + \cancel{3} \times \frac{n(n-1)(n-2)}{1 \times 2 \times \cancel{3}} p^3 q^{n-3} + \ldots\ldots + n \times 1\, p^n q^0$

$= np\left[q^{n-1} + (n-1) p q^{n-2} + \frac{(n-1)(n-2)}{1 \times 2} p^2 q^{n-3} + \ldots\ldots p^{n-1}\right]$

$\left[(a+b)^{n-1} = {}^{n-1}C_0\, a^{n-1} + {}^{n-1}C_1\, a^{n-2} b + \ldots\ldots + {}^{n-1}C_{n-1}\, b^{n-1}\right]$

$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \ldots\ldots + {}^nC_n\, b^n$

Sum of Probability = 1

$\therefore p+q=1$

$= np[q+p]^{n-1}$

$= np \cdot \times 1^{n-1} = \underline{\underline{np}}$

$Var(x) = E(x^2) - [E(x)]^2$

$x^2 = x(x-1)+x$

$E(x^2) = \sum x^2 p(x)$

$= \sum [x(x-1)+x]\, {}^nC_x\, p^x q^{n-x}$

$= \sum x(x-1)\, {}^nC_x\, p^x q^{n-x} + \sum x\, {}^nC_x\, p^x q^{n-x}$

$= 0 + 0 + 2\times 1\, {}^nC_2\, p^2 q^{n-2} + 3\times 2\, {}^nC_3\, p^3 q^{n-3}$

$+ \cdots + n(n-1)\, {}^nC_n\, p^n q^{n-n} + np$

$= 2\times 1 \times \frac{n(n-1)}{1\times 2} p^2 q^{n-2} + 3\times 2\, \frac{n(n-1)(n-2)}{1\times 2\times 3} p^3 q^{n-3}$

$+ \cdots + n(n-1)p^n + np$

$= n(n-1)p^2 [q^{n-2} + (n-2)pq^{n-3} + \cdots + p^{n-2}] + np$

$= n(n-1)p^2 (q+p)^{n-2} + np$

$= n(n-1)p^2 \times (1)^{n-2} + np$

$p+q=1$

$\therefore \underline{\underline{q=1-p}}$

$= n(n-1)p^2 + np$

$Var(x) = n(n-1)p^2 + np - (np)^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = -np^2 + np$

$= np(\underbrace{1-p}_{q}) = \underline{npq}$

Mean of a binomial random variable is np and variance is npq.

Q. Find the binomial distribution for which mean = 5 and variance = $\frac{15}{4}$

Soln: Given $np = 5 \longrightarrow ①$

variance $npq = \frac{15}{4} \longrightarrow ②$

$5q = \frac{15}{4}$ $\quad q = \frac{15}{4 \times 5} = \frac{3}{4}$

$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$n \times \frac{1}{4} = 5 \Rightarrow n = 20$

∴ Binomial distribution is :

$p(x) = {}^{20}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{20-x}$, $x = 0, 1, 2 \ldots 20$

Q. 5 bit secret code words are sent across a noisy channel. Each bit has probability 0.05 of being received in error independently of others. If not more than 2 bits are received in error, the code word can be correctly decoded. What is the probability that a code word received can be correctly decoded?

Soln:

P = Probability that a bit in error = 0.05

$q = 1 - p = 1 - 0.05 = 0.95$

Let $n = 5$, Let x denotes the no of bits in error then x is a binomial variable with pmf

$$p(x) = {}^5C_x (0.05)^x (0.95)^{5-x}, \quad x = 0, 1, 2, \ldots 5$$

Probability that the code word can be correctly decoded

$= 0, 1, 2$ $\qquad \frac{5 \times 4}{1 \times 2}$

$= P(x=0) + P(x=1) + P(x=2)$

$= {}^5C_0 (0.05)^0 (0.95)^{5-0} + {}^5C_1 (0.05)^1 (0.95)^4 +$
${}^5C_2 (0.05)^2 (0.95)^3$

$= (0.95)^5 + 5 \times (0.05)(0.95)^4 + 10 (0.05)^2 (0.95)^3$

Q. A communication setup consists of 5 subsystems each of which may fail independently with probability 0.15. The system will work if at least 3 of the subsystems operate correctly. what is the probability that the system will work? what is the expected number of correctly working subsystems?

Soln:

$p = 0.15 \quad q = 0.85 \quad n = 5$

Let x denotes the no of systems in fail then x is a binomial random variable with pmf ${}^5C_x (0.15)^x (0.85)^{5-x}$

where $x = 0, 1, 2 \ldots 5$

at least 3:

P(system will work) = P(X = 0, 1, 2)

$= {}^5C_0 (0.15)^0 (0.85)^5 + {}^5C_1 (0.15)^1 (0.85)^4 + {}^5C_2 (0.15)^2 (0.85)^3$

$= (0.85)^5 + 5 (0.15)(0.85)^4 + 10 (0.15)^2 (0.85)^3$

The Average no of working subsystem = nq

$= 5 \times 0.85$

Here p is the no of system in fail so in order to find the no of system success we choose q which is P(success)

Q. In sampling a large no of parts manufactured by a machine, the mean no of defectives in a sample of 15 is 2. Out of 1000 such samples how many would be expected to contain

1) No Defective
2) Exactly 2 defective
3) Not more than 2 defectives.

Soln: p = Probability of defectives $= \frac{2}{15}$

$q = 1 - p = 1 - \frac{2}{15} = \frac{13}{15}$, $n = 15$

Let x denote the no of defectives then x is a binomial random variable with pmf

$p(x) = {}^{15}C_x \left(\frac{2}{15}\right)^x \left(\frac{13}{15}\right)^{15-x}$, $x = 0, 1, 2 \ldots 15$

1) $P(\text{No Defective}) = P(X=0)$

$$= {}^{15}C_0 \left(\frac{2}{15}\right)^0 \left(\frac{13}{15}\right)^{15-0} = \left(\frac{13}{15}\right)^{15}$$

out of 1000 such ~~people~~ samples, the no of sample having no defectives $= 1000 \times \left(\frac{13}{15}\right)^{15}$

2) $P(2 \text{ defective}) = P(X=2)$

$$= {}^{15}C_2 \left(\frac{2}{15}\right)^2 \left(\frac{13}{15}\right)^{15-2}$$

Required num $= 1000 \times {}^{15}C_2 \left(\frac{2}{15}\right)^2 \left(\frac{13}{15}\right)^{13}$

3) $P(\text{not more than 2 defective}) = P(X=0,1,2)$

$${}^{15}C_0 \left(\frac{2}{15}\right)^0 \left(\frac{13}{15}\right)^{15-0} + {}^{15}C_1 \left(\frac{2}{15}\right)^1 \left(\frac{13}{15}\right)^{14} + {}^{15}C_2 \left(\frac{2}{15}\right)^2 \left(\frac{13}{15}\right)^{13}$$

Q. out of 1000 families of 4 children each, how many families would you expect ~~but~~ 1) no girls 2) majority girls 3) at most 2 girls.

soln: $P(\text{girl}) = \frac{1}{2}$, $q = \frac{1}{2}$, $n=4$

Let x denote the no of girls then x is a binomial random variable with pmf

$$p(x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \quad x = 0,1,2,3,4$$

1) $p(\text{no girls}) = P(X=0)$

$$= {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4$$

out of 1000 families of 4 children each, the no of families with no girls $= \left(\frac{1}{2}\right)^4 \times 1000 = \frac{1000}{16} = \underline{62.5} = 63$

2) P(majority girls) = $P(x=3,4)$

$\frac{4\times3\times2}{1\times2\times3}$

$= {}^4C_3 \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + {}^4C_4 \left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0$

$= 4\times\frac{1}{2^4} + 1\times\frac{1}{2^4} = 4\times\frac{1}{16} + \frac{1}{16} = \frac{5}{16}$

$= \frac{5}{16}\times 1000 = \frac{5000}{16} = \underline{313}$

$\frac{4\times3}{1\times2}$

3) P(at most 2 girls) = $P(x=0,1,2)$

$= {}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + {}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

$= 1\times\frac{1}{16} + 4\times\frac{1}{16} + 6\times\frac{1}{16}$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$

$\frac{11}{16}\times 1000 = \frac{11000}{16} = \underline{688}$

✱ <u>Fitting of Binomial Distribution</u>

The table given below is the no of thunderstorms reported in a particular summer month by 100 meteorological stations. Fit a binomial distribution and compute the theoretical ~~sequences~~ frequencies.

No of thunderstorm:	0	1	2	3	4	5
No of station:	22	37	20	13	6	2

Soln: Here $n = 5$

Mean $= 0\times22 + 1\times37 + 2\times20 + 3\times13 + 4\times6 + 5\times2$

$= 37 + 40 + 39 + 24 + 10 = \underline{150}$

$= \dfrac{150}{100} = \dfrac{3}{2} = \underline{1.5}$

$$\begin{array}{r} 37 \\ 39 \\ 24 \\ 50 \\ \hline 150 \end{array}$$

For a binomial distribution mean $= np$

$\Rightarrow np = 1.5 \quad \Rightarrow 5p = 1.5 \quad p = \dfrac{1.5}{5} = \underline{0.3}$

$q = 1 - p = 0.7$

$\therefore$ Fitted Binomial distribution is

$p(x) = 5_{C_x}(0.3)^x(0.7)^{5-x}, \quad x = 0, 1, 2 .. 5$

x	freq	theoretical prob $= p(x)$	theoretical freq $= 100 \times p(x)$
0	22	$5_{C_0}(0.3)^0(0.7)^5 = 0.168$	$0.168 \times 100 = 16.8 \approx 17$
1	37	$5_{C_1}(0.3)^1(0.7)^4 =$	
2	20	$5_{C_2}(0.3)^2(0.7)^3$	
3	13	$5_{C_3}(0.3)^3(0.7)^2$	
4	6	$5_{C_4}(0.3)^4(0.7)^1$	
5	2	$5_{C_5}(0.3)^5(0.7)^0$	

Poisson Distribution

A discrete Random variable x is said to follow Poisson distribution with parameter $\lambda > 0$ if its pmf is

$$p(x) = \frac{e^{-\lambda} \lambda^{n}}{x!} , \quad x = 0, 1, 2 \ldots$$

Result: Poisson distribution is the limiting form of binomial distribution in the sense that as n become larger, p approaches 0 and the mean $np \to$ a constant, say λ $(\to)$

Q. Find the mean and variance of poisson distribution

Let x be a poisson random variable then its pmf is

$$p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} , \quad x = 0, 1, 2 \ldots$$

$$\text{Mean} = \sum x \, p(x)$$

$$= 0 \times \frac{e^{-\lambda} \lambda^{0}}{0!} + 1 \times \frac{e^{-\lambda} \lambda^{1}}{1!} + 2 \times \frac{e^{-\lambda} \lambda^{2}}{2!} + 3 \times \frac{e^{-\lambda} \lambda^{3}}{3!} \ldots$$

$$= e^{-\lambda} \lambda \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \ldots \right)$$

$$\left(e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right)$$

$a^m a^n = a^{m+n}$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= e^{-\lambda + \lambda} \lambda = e^{0} \lambda = \underline{\lambda}$$

$Var(X) = E(X^2) - (E(x))^2$

$E(X^2) = \sum x^2 p(x)$

$= \sum [x(x-1)+x] \frac{e^{-\lambda}\lambda^x}{x!}$

$= \sum x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum x \frac{e^{-\lambda}\lambda^x}{x!}$

Given that $x = 0, 1, 2 \ldots$

$= 0 + 0 + 2\times 1 \times \frac{e^{-\lambda}\lambda^2}{2!} + 3\times 2 \frac{e^{-\lambda}\lambda^3}{3!} + \frac{4\times 3\times e^{-\lambda}\lambda^4}{4!} + \ldots + \lambda$

$= e^{-\lambda}\lambda^2 [1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots] + \lambda$

$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda$

$= \lambda^2 e^0 + \lambda = \lambda^2 + \lambda$

$\therefore var(X) = E(X^2) - [E(x)]^2$

$= \lambda^2 + \lambda - (\lambda)^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \underline{\underline{\lambda}}$

Mean(X) = var(X)

Q. An car hire firm has 2 cars which it hires out day by day the no of demands for a car on each day is distributed as Poisson distribution with mean 1.5 cars per day.

Find the probability that on a day :

1) There is no demand

2) Some demand is refused

Soln:

Let x denote the no of demands for a day

Given that x follows poisson distribution with mean

$\lambda = 1.5$

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad x = 0, 1, 2, \ldots$$

$$P(\text{no demand}) = P(x=0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-\lambda}$$

$$= e^{-1.5}$$

$P(\text{some demand refused}) = P(x = 3, 4, 5 \ldots)$ because only 2 cars

$$= 1 - P(x = 0, 1, 2)$$

$$= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

Q. In a Poisson distribution probability corresponding to 3 successes is 4/5 times probability corresponding to 4 successes. find the mean and SD of a distribution.

Soln:

Given $p(x=3) = \frac{4}{5}\, p(x=4)$

$$\frac{\cancel{e^{-\lambda}} \lambda^3}{3!} = \frac{4}{5} \frac{e^{-\lambda} \lambda^4}{4!}$$

$$\frac{1}{\cancel{1} \times \cancel{2} \times \cancel{3}} = \frac{4}{5} \times \frac{\lambda}{1 \times 2 \times 3 \times 4}$$

$\underline{\underline{\lambda = 5}}$

Mean $= \lambda = 5$ Variance $= \lambda = 5$ SD $= \sqrt{\text{Variance}} = \sqrt{5}$

Q. B/w the hours of 2 pm and 4 pm the average no of phone calls per minute coming into the company is 2.5. find the probability that during one particular minute there will be :

1) no phone calls at all.

2) at least 2 calls.

Soln: Let x denotes the no of phone calls

we assume that x follows poisson distribution with $\lambda = 2.5$.

$$p(x) = \frac{e^{-2.5} (2.5)^x}{x!} \quad \text{where } x = 0, 1, 2, 3 \ldots$$

1) $P(x=0) = \dfrac{e^{-2.5} (2.5)^0}{0!} = e^{-2.5}$

2) $P(\text{at least 2 phone calls}) = p(2, 3, 4 \ldots)$

$= 1 - P(x = 0, 1)$

$$= 1 - \left[\frac{e^{-2.5} (2.5)^0}{0!} + \frac{e^{-2.5} (2.5)^1}{1!} \right]$$

$= 1 - [e^{-2.5} + 2.5 e^{-2.5}]$

$= 1 - 3.5 e^{-2.5}$

Q: Fit a poisson distribution for the following data

x	0	1	2	3	4
f	63	28	6	2	1

also find the theoretical frequencies

28
22
0

Soln:

mean $= 0 \times 63 + 1 \times 28 + 2 \times 6 + 3 \times 2 + 4 \times 1$

$= \frac{50}{100} = \underline{0.5}$

$f = 100$

For a Poisson distribution mean $= \lambda$

$\therefore \lambda = 0.5$

Thus the fitted Poisson distribution is

$$p(x) = \frac{e^{-0.5} (0.5)^2}{x!}, \quad x = 0, 1, 2 \ldots$$

x	f	Theoretical prob $= p(x)$	Theoretical freq $= 100 \times p(x)$
0	63	$p(x) = \frac{e^{-0.5}(0.5)^0}{0!} = 0.6065$	$100 \times 0.6065 = 60.65 \approx 61$
1	28	$p(x) = \frac{e^{-0.5}(0.5)^1}{1!} = 0.3032$	$100 \times 0.3032 = 30.32 \approx 30$

Discrete Uniform Distribution

A random variable x is said to follow a discrete uniform distribution if its pmf is.

$$p(x) = \frac{1}{n}, \quad x = 1, 2, \dots n$$

Q. Find the mean and variance of discrete uniform distribution

Soln:

The pmf of a discrete uniform distribution is given by

$$p(x) = \frac{1}{n}, \quad x = 1, 2, \dots n$$

Mean $= E(x)$

$$= \sum x\, p(x)$$

$$= 1 \times \frac{1}{n} + 2 \times \frac{1}{n} + 3 \times \frac{1}{n} + \dots + n \times \frac{1}{n}$$

$$= \frac{1}{n}[1 + 2 + 3 + \dots + n]$$

$$= \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$$

variance $= E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 \cdot p(x)$$

$$= 1^2 \times \frac{1}{n} + 2^2 \times \frac{1}{n} + 3^2 \times \frac{1}{n} + \dots + n^2 \times \frac{1}{n}$$

$$= \frac{1}{n}(1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

$$\text{Var} = \frac{(n+1)(2n+1)}{6} - \left(\frac{(n+1)}{2}\right)^2$$

$$= \cancel{\frac{3(n+1)(2n+1) - 3(n+1)}{24}} \quad \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$= \frac{4(2n^2+3n+1) - 6(n^2+2n+1)}{24}$$

$$= \frac{8n^2+12n+4-6n^2-12n-6}{24}$$

$$= \frac{2n^2-2}{24} = \frac{2(n^2-1)}{24} = \frac{n^2-1}{12}$$

Q. If x is a discrete random variable following uniform distribution where variance is 2, find mean of x. If $y = 3x+2$, find $E(y)$ and $\text{var}(y)$

Soln:

Given $\frac{n^2-1}{12} = 2$

$n^2 - 1 = 24 \Rightarrow n^2 = 25 \Rightarrow \underline{n = 5}$

$\text{mean} = \frac{n+1}{2} = \frac{5+1}{2} = \underline{3}$

$y = 3x + 2$

$E(Y) = E(3x + 2)$

$= 3E(x) + 2$ $\quad (\because E(x) = \text{mean} = 3)$

$= 3 \times 3 + 2 = \underline{11}$

$var(y) = var(3x + 2)$ $\quad E(ax+b) = aE(x) + b$

$= 3^2 \cdot var(x)$ $\quad var(ax+b) = a^2 var(x)$

$= 9 \times 2 = \underline{18}$

Geometric Distribution

A discrete random variable x is said to follow geometric distribution with parameters p and q. If its pmf is

$$p(x) = pq^x, \quad x = 0, 1, 2, \ldots \quad 0 < p \le 1, \; p + q = 1$$

Interpretation

Consider an experiment with only 2 possible outcomes say success and failure with probabilities p and q respectively. Suppose the experiment is repeated until success occurs. Let x denotes the no of failures before the success then x can take the values $0, 1, 2 \ldots$

$p(X = x) = p(\text{getting the outcome } \overbrace{FFF\ldots\ldots}^{x \text{ failure}} S)$

$= P(F)P(F)\ldots\ldots P(F)P(S)$

$= \underline{q^x p}$

Q. A probability that an applicant for a drivers licence will pass the road test on any given trial is 0.8. what is the probability that he will finally pass the test?

i) on the 4th trial

2) In fewer than 4 trials

Soln:

$p = 0.8 \quad q = 0.2$

Let x denote the no of failures before success then it follows Geometric distribution

$p(x) = pq^x = (0.8)(0.2)^x$

i) p(on getting licence on 4th trial) = $p(x=3)$

$= (0.8)(0.2)^3 = 0.8 \times 0.008 = \underline{0.064}$

2) $p(x < 3) = p(x = 0, 1, 2)$

Q. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot in any one shot is 0.7, what is the probability that i) the target would be hit on the tenth attempt 2) It takes him less than 4 shots

$p = 0.7 \quad q = 0.3$

Let x denote the no of failures before the first hit then x follows the Geometric distribution with pmf